Construction of bilinear control Hamiltonians using the series product and quantum feedback

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Abstract

We show that it is possible to construct closed quantum systems governed by a bilinear Hamiltonian depending on an arbitrary input signal. This is achieved by coupling the system to a quantum input field and performing a feedback of the output field back into the system to cancel out the stochastic effects, with the signal being added to the field between these events and later subtracted. Here we assume the zero time delay limit between the various connections and operations.

1 Introduction

The theory of quantum control can be divided into open loop and closed loop problems. In open loop problems a common situation is to investigate various notions of controllability for closed systems governed by a bilinear Hamiltonian, that is, taking the form

$$H(\mathbf{u}) = H_0 + \sum_k H_k u_k \tag{1}$$

where the H_0, H_k are self-adjoint and the u_k are real-valued functions of time called the controls [1],[4]. Closed loop problems can be divided into two classes: *measurement-based control* which requires feedback from a controller based on quantum state filtering using the results of a measurement of the output [5]-[6]; *coherent control* where the feedback is fully quantum and no measurement is performed [7]-[9]. Typically in closed loop problems, the information is carried from system to apparatus/controller by quantum input processes [10] and so the system undergoes a stochastic open dynamical evolution. A theory for forming feedforward and feedback of quantum input-output systems has been developed recently where the concept of the series product for determining the generators of systems in series was introduced [11]. This is part of a more general theory which extends to (indirect) feedback loops mediated through beam splitters [12].

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In this paper, we wish to describe a coherent control strategy which allows for a *feedback cancellation* of the stochastic component of the dynamics. Here the input noise is passed through the system, has a signal added, is passed through the system again to undo the stochastic component of the evolution, and finally has the original signal subtracted. Assuming that these operations occur without time delays, the overall result is a *closed* system dynamics with a modified Hamiltonian which *generically* is bilinear.

2   Input-Output Devices

In a Markov model of an open quantum mechanical system interacting driven by a bosonic field in the vacuum state, we introduce input processes $b_i(t)$ for $i = 1, \cdots, n$ with the canonical commutation relations, [10],

$$[b_i(t), b_j^\dagger(s)] = \delta_{ij}\,\delta(t-s). \tag{2}$$

It is convenient to write these as a column vector of length n

$$\mathbf{b}^{\text{in}}(t) = \begin{pmatrix} b_1(1) \\ \vdots \\ b_n(t) \end{pmatrix}. \tag{3}$$

We sketch the system plus field as a two port device having an input and an output port.
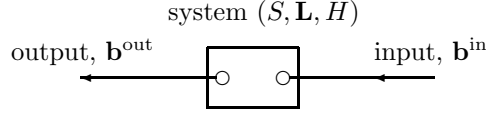

Figure 1: input-output component

The evolution can be described by the unitary process $\{V_t\}$ satisfying the Wick-ordered (creators appear on the left, annihilators on the right) differential equation of the form

$$
\begin{aligned}
\frac{d}{dt}V_t &= \mathbf{b}^{\text{in}}(t)^\dagger (S(t)-I)V_t\mathbf{b}^{\text{in}}(t) + \mathbf{b}^{\text{in}}(t)^\dagger \mathbf{L}(t)V_t \\
&\quad - \mathbf{L}^\dagger(t)S(t)V_t\mathbf{b}^{\text{in}}(t) - (\frac{1}{2}\mathbf{L}^\dagger(t)\mathbf{L}(t) - iH(t))V_t,
\end{aligned}
$$

with initial value $V_0 = I$. The coefficients $S_{ij}(t)$, $L_i(t)$ and $H(t)$ are operators having the property that they are adapted (i.e., commute with the fields $b_j(s)$ and $b_j^\dagger(s)$ for earlier times $s < t$, and such that $S(t) = (S_{ij}(t))$ is a unitary matrix with operator entries (i.e., $\sum_k S_{ik}S_{jk}^\dagger = \delta_{ij} = \sum_k S_{ki}^\dagger S_{kj}$), $\mathbf{L}(t)$ in the column vector of operators $(L_i(t))$, and H is self-adjoint. This equation can be

interpreted as a quantum stochastic differential equation [10], [13]. The solution is known to exist and be unique, and we shall denote it as

$$V_t = V_t \left(S, \mathbf{L}, H; \mathbf{b}^{\text{in}} \right) \tag{4}$$

Let X be a fixed operator of the system and set $\tilde{X}(t) \triangleq V_t^\dagger X V_t$, then we obtain the Heisenberg-Langevin equation

$$\frac{d}{dt}\tilde{X}(t) = \mathcal{V}_t \left(X; S, \mathbf{L}, H; \mathbf{b}^{\text{in}} \right)$$
$$\equiv \mathbf{b}^{\text{in}}(t)^\dagger \, \mathcal{S}(X;t) \, \mathbf{b}^{\text{in}}(t) + \mathbf{b}^{\text{in}}(t)^\dagger \, \mathcal{J}(X,t) + \mathcal{K}(X,t) \, \mathbf{b}^{\text{in}}(t) + \mathcal{L}(X,t),$$

where we encounter the super-operators

$$\mathcal{S}(X;t)_{ij} = \sum_k \tilde{S}_{ki}^\dagger \tilde{X} \tilde{S}_{kj} - \tilde{X} \delta_{ij},$$

$$\mathcal{J}(X,t)_i = \sum_j \tilde{S}_{ji}^\dagger [\tilde{X}, \tilde{L}_j], \ \mathcal{K}(X,t)_j = \sum_i [\tilde{L}_i^\dagger, X] \tilde{S}_{ij},$$

$$\mathcal{L}(X,t) = \frac{1}{2} \sum_i \tilde{L}_i^\dagger [\tilde{X}, \tilde{L}_i] + \frac{1}{2} \sum_i [\tilde{L}_i^\dagger, \tilde{X}] \tilde{L}_i - i[\tilde{X}, \tilde{H}].$$

Here $\tilde{S}_{ij}(t)$ denotes $V_t^\dagger S_{ij}(t) V_t$, etc. The final term is a (time-dependent) Lindbladian. In the case $S = 1$, this equation reduces to the Heisenberg-Langevin equations introduced by Gardiner and Collett [10]. The output fields are defined by $b_i^{\text{out}}(t) = V_t^\dagger b_i(t) V_t$ and we have the input-output relation

$$\mathbf{b}^{\text{out}}(t) = \tilde{S}(t) \mathbf{b}^{\text{in}}(t) + \tilde{\mathbf{L}}(t), \tag{5}$$

that is, $b_i^{\text{out}}(t) = \sum_{j=1}^n V_t^\dagger S_{ij}(t) V_t b_j(t) + V_t^\dagger L_i(t) V_t$.

2.1 Examples

2.1.1 Cavity Mode

For a cavity mode a and complex damping $\kappa = \frac{1}{2}\gamma + i\omega$ with a single input field, we have

$$V_t = V_t \left(I, \sqrt{\gamma} a, \omega a^\dagger a; b^{\text{in}} \right).$$

Here $b^{\text{out}}(t) = b^{\text{in}}(t) + \sqrt{\gamma} V_t^\dagger a V_t$.

2.1.2 Pure Hamiltonian, HAM(H)

The situation of a closed system is described by the device HAM(H),

$$V_t = V_t \left(I, 0, H; \mathbf{b}^{\text{in}} \right)$$

and there is no interaction between the system and the input fields. For H time-independent, we have $V_t \equiv \exp\{-itH\}$.

2.1.3 Beam Splitter, BS (T)

We take $S = T$ a unitary matrix with c-number entries. The beam splitter with matrix T is the device BS (T) described by

$$V_t = V_t \left(T, 0, 0; \mathbf{b}^{\text{in}} \right).$$

The system dynamics is trivial nd we have the input-output relation

$$\mathbf{b}^{\text{out}} (t) = T\mathbf{b}^{\text{in}} (t). \tag{6}$$

2.1.4 Signal Adding Devices, ADD $(\mathbf{u})$

Let $\mathbf{u} = (u_j)$ be a square-integrable function of time taking values in $\mathbb{C}^n$. We consider the device with dynamics

$$V_t = V_t \left(I, \mathbf{u}, 0; \mathbf{b}^{\text{in}} \right).$$

This has trivial system dynamics and input-output relation

$$\mathbf{b}^{\text{out}} (t) = \mathbf{b}^{\text{in}} (t) + \mathbf{u} (t).$$

Here we think of $\mathbf{u}$ as a signal carried by the input field. Alternatively we may think of the field as now being in the coherent state with intensity $\mathbf{u}$. We shall refer to such a component *device adding a signal* $\mathbf{u}$ to the field. Such a device will be denoted as ADD $(\mathbf{u})$.

3 Systems in Series

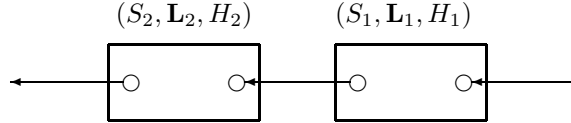Let us consider two systems in cascade as shown below.

figure 2: Cascaded systems

The time delay for the output of the first system $(S_1, \mathbf{L}_1, H_1)$ to reach the second system $(S_2, \mathbf{L}_2, H_2)$ as input will be some $\tau > 0$, and we are interested in the limit situation where $\tau \to 0$. This leads to a single effective Markovian model with coefficients given by the *series product* [11]

$$(S_2, \mathbf{L}_2, H_2) \lhd (S_1, \mathbf{L}_1, H_1) \triangleq \left(S_2 S_1, \mathbf{L}_2 + S_2 \mathbf{L}_1, H_1 + H_2 + \text{Im} \mathbf{L}_2^\dagger S_2 \mathbf{L}_1 \right). \tag{7}$$

We remark that the series product is valid even when the observables of the two systems are not suppose to commute, that is, we have feedback into the same system. Figure 3 below shows a direct feedback situation with an equivalent model given by the series product.
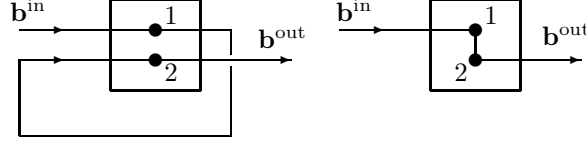
Figure 3: Direct feedback, $S_2 \lhd S_1$.

The series product is generally not commutative, but is associative. Pure Hamiltonian devices can be entered in series at any point due to the identity

$$(S, \mathbf{L}, H) \lhd (I, 0, H_0) = (S, \mathbf{L}, H + H_0) = (I, 0, H_0) \lhd (S, \mathbf{L}, H). \qquad (8)$$

3.1 Derivation of the Series Product

A rigorous derivation of the series product is given in [11], [12], however, we now give an alternative heuristic derivation extending the argument originally used by Gardiner [15]. For convenience we work withonly one noise field and supress the time variable. The Heisenberg-Langevin equation for any observable X of the joint system will be

$$\frac{d}{dt}\tilde{X}(t) = \sum_{\alpha=1,2} \mathcal{V}_t \left(X; S_\alpha, L_\alpha, H_\alpha; b_\alpha^{\mathrm{in}} \right)$$

where $b_1^{\mathrm{in}}(t)$ is the overall input $b^{\mathrm{in}}(t)$ and

$$b_2^{\mathrm{in}}(t) = b_1^{\mathrm{out}}\left(t - 0^+\right) \equiv \tilde{S}_1(t) b^{\mathrm{in}}(t) + \tilde{L}_1(t).$$

The super-operators correspond to the coefficients for the first and second system respectively for $\alpha = 1, 2$. We eliminate the fields $b_\alpha^{\mathrm{in}}(t)$ and write in terms of $b^{\mathrm{in}}(t)$. After some algebra we find that the Langevin equation has the form (**??**) with

$$
\begin{aligned}
\mathcal{S}(X) &= \mathcal{S}_1(X) + \tilde{S}_1^\dagger \mathcal{S}_2(X)\tilde{S}_1 \equiv (\tilde{S}_2\tilde{S}_1^\dagger)\tilde{X}(\tilde{S}_2\tilde{S}_1) - \tilde{X} \\
\mathcal{J}(X) &= \mathcal{J}_1(X) + \tilde{S}_1^\dagger \mathcal{J}_2(X) + \tilde{S}_1^\dagger \mathcal{S}_2(X)\tilde{L}_1 \equiv (\tilde{S}_2\tilde{S}_1)^\dagger \left[\tilde{X}, \tilde{L}_2 + \tilde{S}_2\tilde{L}_1\right], \\
\mathcal{K}(X) &= \mathcal{K}_1(X) + \mathcal{K}_2(X)\tilde{S}_2 + \tilde{L}_1^\dagger \mathcal{S}_2(X)\tilde{S} \equiv [(\tilde{L}_2 + \tilde{S}_2\tilde{L}_1)^\dagger, \tilde{X}](\tilde{S}_2\tilde{S}_1) \\
\mathcal{L}(X) &= \mathcal{L}_1(X) + \mathcal{L}_2(X) + \tilde{L}_1^\dagger \mathcal{J}_2(X) + \mathcal{K}_2(X)\tilde{L}_1 + \tilde{L}_1^\dagger \mathcal{S}_2(X)\tilde{L}_1
\end{aligned}
$$

and by inspection we deduce that this is of the standard form with $S \equiv S_2 S_1$, $L \equiv L_2 + S_2 L_1$. After some algebra we show that $\mathcal{L}(X)$ is a Lindbladian with coupling operator $\tilde{L}$ and Hamiltonian $\tilde{H} \equiv \mathrm{Im}\tilde{L}_2^\dagger \tilde{S}_2 \tilde{L}_1$.

3.2 Pre- and Post-applications of beam splitters

Given a fixed component with coefficients $(S, \mathbf{L}, H)$, we apply a beam splitter T to the inputs prior to entry and the inverse T^{-1} to outputs after exit from

5

the component. Assuming zero time delay in travelling from the beam splitters to the components, we have for the combined system

$$\left(T^{-1}, 0, 0\right) \lhd (S, \mathbf{L}, H) \lhd (T, 0, 0) = \left(T^{-1}ST, T^{-1}\mathbf{L}, H\right). \tag{9}$$

We note the identity

$$V_t\left(S, \mathbf{L}, H; T\mathbf{b}^{\text{in}}\right) \equiv V_t\left(T^{-1}ST, T^{-1}\mathbf{L}, H; \mathbf{b}^{\text{in}}\right),$$

which means that the combined system could be viewed as the original system, but driven by "rotated" input $T\mathbf{b}^{\text{in}}$.

4 Bilinear Hamiltonians

4.1 Constructions with Noise

Let us fix a matrix T of c-numbers. We pass the initial input through a device adding a signal $\mathbf{v} = T^{-1}\mathbf{u}$, and then pass the output as input through a general component with coefficients $(T, \mathbf{L}, H)$. In the zero-delay limit, the combined system is determined by

$$(T, \mathbf{L}, H_0) \lhd \left(I, T^{-1}\mathbf{u}, 0\right) = \left(T, \mathbf{L} + \mathbf{u}, H_0 + \text{Im}\mathbf{L}^\dagger\mathbf{u}\right).$$

We may subsequently pass the output through a third component which adds the signal $-\mathbf{u}$, we find

$$(I, -\mathbf{u}, 0) \lhd (T, \mathbf{L}, H_0) \lhd \left(I, T^{-1}\mathbf{u}, 0\right) = \left(T, \mathbf{L}, H_0 + 2\text{Im}\mathbf{L}^\dagger\mathbf{u}\right).$$

The result is that we modify the Hamiltonian to

$$H\left(\mathbf{u}\right) = H_0 + 2\text{Im}\mathbf{L}^\dagger\mathbf{u} = H_0 + 2\sum_j \left(L_{j,R}u_{j,I} - L_{j,I}u_{j,R}\right) \tag{10}$$

where $u_j = u_{j,R} + iu_{j,I}$ and $L_j = L_{j,R} + iL_{j,I}$ with $u_{j,R}, u_{j,I}$ real and $L_{j,R}, L_{j,I}$ self-adjoint.

The combined model includes a coupling via the operator $\mathbf{L}$ to the environment so that the evolution, given by $V_t\left(T, \mathbf{L}, H\left(\mathbf{u}\right), \mathbf{b}^{\text{in}}\right)$. To counteract this now stochastic evolution we may perform a homodyne measurement on a quadrature of $\mathbf{b}^{\text{out}}$ and employ a filter to estimate the quantum state of the system conditioned on the measurement record [5].

4.2 Constructions without Noise

We now show how, given a fixed system with internal Hamiltonian H_0 and which can couple to input noise with operators $\mathbf{L}$, to construct an effectively closed system with internal Hamiltonian $H\left(\mathbf{u}\left(t\right)\right)$ for prescribed signal $\mathbf{u}$, with bilinear Hamiltonian $H\left(\mathbf{u}\right) = H_0 + \text{Im}\mathbf{L}^\dagger\mathbf{u}$. We refer to this general procedure as *quantum noise cancellation by feedback*. The system has been engineered so

that it interacts with the input noise, but this is then undone by feedback of the output noise back into the system. Between these events, we shall add in the signal term which we later subtract out.

Let $\mathtt{SYS}(\mathbf{L})$ denote the device with coefficients $(I, \mathbf{L}, 0)$, then what we shall show is that we can obtain the closed system $\mathtt{HAM}(H(\mathbf{u}))$ from only the devices $\mathtt{SYS}(\mathbf{L})$, $\mathtt{ADD}(\pm\mathbf{u})$, $\mathtt{BS}(-I)$, and $\mathtt{HAM}(H_0)$ in series.

Using identity (9) with $T = -I$ we have

$$(-I, 0, 0) \lhd (I, \mathbf{L}, 0) \lhd (-I, 0, 0) = (I, -\mathbf{L}, 0), \tag{11}$$

or $\mathtt{BS}(-I) \lhd \mathtt{SYS}(\mathbf{L}) \lhd \mathtt{BS}(-I) = \mathtt{SYS}(-\mathbf{L})$.

Now $\mathtt{ADD}(\mathbf{u}) \lhd \mathtt{SYS}(-\mathbf{L}) \lhd \mathtt{ADD}(-\mathbf{u}) = \mathtt{SYS}(\mathbf{L})$, that is,

$$(I, \mathbf{u}, 0) \lhd (I, -\mathbf{L}, 0) \lhd (I, -\mathbf{u}, 0) = \left(I, -\mathbf{L}, \mathrm{Im}\mathbf{L}^\dagger\mathbf{u}\right)$$

and this can be used to exactly negate the $\mathbf{L}$-coupling:

$$\left(I, -\mathbf{L}, \mathrm{Im}\mathbf{L}^\dagger\mathbf{u}\right) \lhd (I, \mathbf{L}, 0) = \left(I, 0, \mathrm{Im}\mathbf{L}^\dagger\mathbf{u}\right).$$

The sequence of output-to-input connections is then given by

$$\mathtt{ADD}(\mathbf{u}) \lhd \mathtt{BS}(-I) \lhd \mathtt{SYS}(\mathbf{L}) \lhd \mathtt{BS}(-I) \lhd \mathtt{ADD}(-\mathbf{u}) \lhd \mathtt{SYS}(\mathbf{L}) = \mathtt{HAM}\left(\mathrm{Im}\mathbf{L}^\dagger\mathbf{u}\right).$$

From (8) we may include the pure Hamiltonian device $\mathtt{HAM}(H_0)$ at any stage in the series, in particular, $\mathtt{HAM}(H_0) \lhd \mathtt{HAM}\left(\mathrm{Im}\mathbf{L}^\dagger\mathbf{u}\right) = \mathtt{HAM}\left(H_0 + \mathrm{Im}\mathbf{L}^\dagger\mathbf{u}\right)$.

The construction requires the introduction of noise through coupling to the input field, but relies on a cancellation of the stochastic component of the dynamics by passing the output field through the system a second time. To achieve the cancellation, we needed to reverse the sign of the coupling operators $\mathbf{L}$ which is achieved by (11). As the noise is being passed through the same physical system twice, we have an example of feedback, however, as we have mentioned, the series product covers this situation.

4.3 Physical Example

As a possible model application we can consider the all-optical feedback experiment proposed by Wiseman and Milburn, [14] section II.B. Here a cavity mode with annihilator a is contained between two mirrors. The input is an external light beam which is shone on the first mirror where it is reflected. The reflection of the mirror induces an interaction with the cavity mode $\left(L = \sqrt{\gamma}a\right)$ and is the reflected beam picks up a phase $e^{i\theta}$. The reflected beam is thenshone on the second mirror again interacting with the cavity mode in the same way. The series product has been previously used to rederive model specification of the resulting model [11], section IV.A.

We now propose a modification where the signal is added to the light beam on route from being reflected off the first mirror to impinging on the second, and then subtracted from the light reflected from the second mirror. We additionally assume that $\theta = \pi$ so that when the beam is relected off either cavity mirror it

picks up a sign change. Suppose that the internal Hamiltonian is $H_0 = \omega_0 a^\dagger a$ and $a(t) = V_t^\dagger a V_t$ is the mode in the Heisenberg picture, then the Heisenberg-Langevin equation will be

$$\dot{a}(t) = -i\omega_0 a(t) - \frac{\sqrt{\gamma}}{2} u(t)$$

corresponding to evolution under the Hamiltonian $\omega_0 a^\dagger a + \frac{\sqrt{\gamma}}{2i} \left(a^\dagger u(t) - a u^*(t) \right)$. This is the familiar situation of a driven harmonic oscillator with control function u [16] where it is known that the vacuum may only be steered to a coherent state.

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